A theory of numerosity for infinite sets where the whole is larger than the part

Mauro Di Nasso – Università di Pisa – Italia

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Summary

1. Introduction: measuring the size of infinite collections.
2. The *Part-Whole Principle* and the idea of *equinumerosity* grounded on Euclid's common notions.
3. *Numerosities* in different contexts:
   - Point sets over the natural numbers (asymptotic numerosities).
   - Point sets over the real line.
   - Sets in a "mathematical universe".
4. Foundational issues and relationships with *nonstandard analysis* and *Ramsey theory*.
5. *Numerosities* and *measures*: the example of Lebesgue measure as a ratio of numerosities.

[Joint research with V. Benci and M. Forti]

Introduction

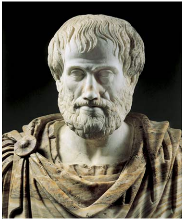
Exploring the idea of infinity can be found throughout the whole history of thought, especially in mathematics.

The idea of a potential infinite as a "never ending process" was already accepted by the ancient Greeks.

$$1,\ 2,\ 3,\ 4,\ 5,\ \quad \dots \quad ,n,\ n+1,\ \dots$$

Aristotle

For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different. ("Physics", IV century BC.)



Euclid

Prime numbers are more than any assigned magnitude of prime numbers.
("Elements", c. 300 BC.)

Throughout the Middle Ages, the idea of an actual infinite was rejected by scholastic philosophers as contradictory. Their motto was the Aristotelian view:

Infinitum actu non datur.

However, recent studies revealed the existence of views on the possibility of comparing infinites, already in the 13th century. (See P. Mancosu, *The Wilderness of Infinity: Robert Grosseteste, William of Auvergne and Mathematical Infinity in the Thirteenth Century*, Oxford University Press, forthcoming.)

Grosseteste

Some infinites are more than others and others less than others; thus, the aggregatio of all numbers, both odd and even, is infinite and it is greater than the aggregatio of all the even numbers, even though this too is infinite, for it exceeds it by the aggregatio of all the odd numbers. ("De Luce", 1225.)

In modern mathematics, it took time to accept the idea of actual infinite as a completed infinite collection.

Gauss, 1831

I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction. (Letter to Schumacher, 1831.)

Things changed when the idea of a "set theory" emerged in the second half of the XIX century.

Bolzano

A multitude which is larger than any finite multitude, i.e., a multitude with the property that every finite set is only a part of it, I will call an infinite multitude.
("Paradoxes of the Infinite", published posthumously 1851)

Disclaimer

While modern mathematics generally accepts actual infinity as standard, a number of mathematicians, philosophers, and logicians have questioned its legitimacy or necessity.

In P. Vopěnka's view, an actual infinite totality cannot be experienced, constructed, or even fully imagined. One can deal with infinity as a process that never ends, but one cannot have an infinite totality as an object.

Other relevant criticisms to actual infinite collections were posed by L. Brouwer (*intuitionism*), H. Weyl (*predicativism*), E. Bishop (*constructivism*) and several other thinkers.

If actual infinite collections do exist, can one measure their size?

A possible view:

- *All infinite collections have the same size, as there is only one infinite.*

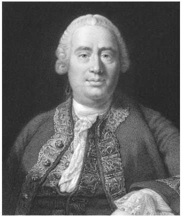
The mathematical idea that different infinite collections may have different sizes was revolutionary.

The goal of extending the notion of size from finite sets to infinite sets was successfully achieved by Georg Cantor with his beautiful theory of cardinal numbers (developed starting around 1886).

Cantor grounded his theory on the notion of *equipotency*, which formalizes a principle commonly attributed to Hume:

Hume

When two numbers are so combined, as that the one has always a unit answering to every unit of the other, we pronounce them equal.
("A Treatise of Human Nature", 1739.)



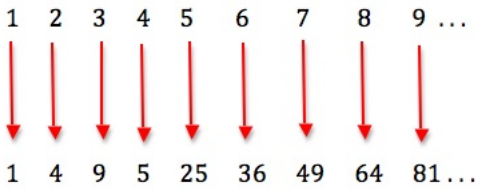
Cantor's definition

Two sets have the same "size" (are equipotent) if and only if they can be put in a one-to-one correspondence.

The above definition extends the usual notion of size for finite sets, and has a clear intuitive content.

However, taking "equipotency" as "having the same size" leads to paradoxical consequences.

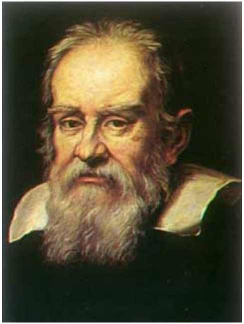
Historically relevant is Galileo's paradox
(*Two New Sciences*, 1638):



Numbers and squares are in a one-to-one correspondance, but squares are a proper subset!

Galileo

So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes "equal", "greater", and "less", are not applicable to infinite, but only to finite, quantities. ("Two New Sciences", 1638.)



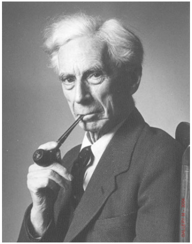
The point is that equipotency for sets collides with the following classical principle for magnitudes, explicitly stated in Euclid's *Elements* as one of the five "Common Notions".

Part-Whole principle

The whole is greater then the part.

Russell

The possibility that whole and part may have the same number of terms is, it must be confessed, shocking to common sense.
("Principles of Mathematics", 1903.)



The notion of numerosity

In this talk I will address the following

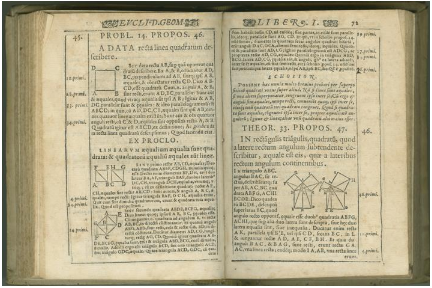
QUESTION:

- Is the Cantorian notion of cardinality the only way to generalize (in a coherent manner) the notion of size of finite sets to infinite sets?
- In particular, can Cantorian cardinality be refined in order to save the Part-Whole principle?
- In particular, can Cantorian cardinality can be refined so that the resulting "numerosities" generalize the natural numbers while maintaining their nice algebraic properties?

A positive answer is given by the theory of numerosity.

In search of a source where the principles for a good theory of magnitudes are explicitly considered, let's go back to the roots and consider the most influential work in the history of mathematics:

Euclid's Elements (c. 300 BC.)



It is worth recalling that until the beginning of the XX century, knowledge of at least the first books of *Euclid's Elements* has always been a fundamental part of the curriculum of university students of all disciplines.

Euclid's Common Notions

At the beginning of his *Elements*, Euclid put a list of postulates, definitions and common notions, as assumptions to be readily accepted by all.

In particular, the common notions describe the fundamental characteristics of any notion of *magnitude*.

Euclid's Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equals be added to equals, the wholes are equal.
3. And if equals be subtracted from equals, the remainders are equal.
4. Things coinciding with [applying onto] one another are equal to one another.
5. The whole is greater than the part.

With the language of contemporary mathematics, we now try to "translate" Euclid's common notions into principles for an *equinumerosity* relation $\approx$ on sets.

(1) Things equal to the same thing are also equal to one another.

$$A \approx C \;\&\; B \approx C \implies A \approx B$$

With the obvious assumption of *reflexivity*, we translate it into the following principle:

(Equivalence)

Equinumerosity $\approx$ is an equivalence relation.

(2) And if equals be added to equals, the wholes are equal.

(3) And if equals be subtracted from equals, the remainders are equal.

There exist natural notions of *sum* and *subtraction* that are compatible with magnitudes.

In the framework of sets, sum and subtraction naturally correspond to disjoint union and relative complement, respectively.

(Sum)

$$A \approx A' \ \& \ B \approx B' \implies A \cup B \approx A' \cup B'$$

whenever $A \cap B = A' \cap B' = \emptyset$.

(Subtraction)

$$A \approx A' \ \& \ B \approx B' \implies A \setminus B \approx A' \setminus B'$$

whenever $A \supseteq B$ and $A' \supseteq B'$.

Note that while the **(Sum)** principle is satisfied by Cantorian cardinality, **(Subtraction)** fails badly.
For instance, for all infinite cardinals $\nu \leq \kappa$, there are sets $A \supset B$ with $|A| = |B| = \kappa$ and $|A \setminus B| = \nu$.

(4) Things coinciding with [applying onto] one another are equal to one another.

. . . the [fourth] Common Notion . . . is intended to assert that superposition is a legitimate way of proving the equality of two figures . . . or . . . to serve as an axiom of congruence.

(T.L. Heath's commentary to Euclid's Elements, 1908.).

Problem of isolating a suitable class of "congruences".

- *Cantor:* Congruences = Bijections.

To keep our framework as general as possible, we do not postulate any specific class of congruences in the context of sets, with only one exception on Cartesian products (which we will see later).

Warning! No transformation T with an *infinite orbit* $\mathcal{O} = \{x, Tx, T^2x, \ldots\}$ can be a congruence because $T(\mathcal{O}) \subsetneq \mathcal{O}$, contradicting the *Whole-Part principle*.

The order of magnitudes

It is implicitly assumed in Euclid's *Common Notions* that magnitudes are arranged in an *order* $\prec$. Clearly, the order must be compatible with the equivalence relation of "having the same magnitude". Moreover, homogeneous magnitudes are traditionally thought as *linearly ordered*.

So, we come to the following principles.

(Order)

Let $A \approx A'$ and $B \approx B'$. Then $A \prec B \iff A' \prec B'$.

(Linearity)

For all A and B, exactly one of the following holds:

$$A \prec B\,, \quad A \approx B\,, \quad B \prec A$$

- In Zermelo-Fraenkel set theory, the property that Cantorian cardinalities are linearly ordered is a non-trivial issue; in fact, in ZF that property is equivalent to the *Axiom of Choice*.

Most relevant to our purposes is the fifth Common Notion:

(5) The whole is greater than the part.

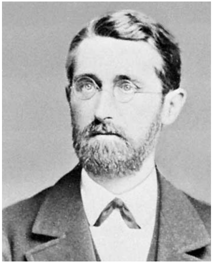
In the framework of set theory, that principle is interpreted as follows:

(Part-Whole)

$$A \subsetneq B \implies A \prec B$$

Cantorian theory of cardinalities strongly violates **(Part-Whole)**. In fact, the following characterization holds, which was used by R. Dedekind as the definition of an infinite set.

- A set is infinite if and only if it has *proper* subsets of the same cardinality.



Given an "equinumerosity" relation, there is a simple and natural way to connect the *order* and the *Part-Whole principle*.

(Superset)

$A \prec B$ if $A' \approx B$ for some proper superset $A' \supsetneq A$.

It is worth noting that the following analogous property could also be considered:

(Subset)

$A \prec B$ if $A \approx B'$ for some $B' \subsetneq B$.

It turns out that **(Subset)** is strictly stronger than **(Superset)**, and is a much more restrictive and difficult property to obtain. Note in fact that there is a "limited" amount of possible subsets, while there is an "unlimited" amount of supersets.

Multiplication of magnitudes

While addition, subtraction and order for magnitudes are explicit in ancient mathematics, multiplication is not considered.

In fact, the geometric idea of a *product* produces objects of a higher dimension.
E.g. product of two lines $\rightsquigarrow$ rectangle.

We remark that measuring the size of *non-homogeneous* magnitudes in geometry by means of the same numbers (*i.e.*, the real numbers $\mathbb{R}$) is a modern point of view.

In a general set theoretic context, it seems natural to consider sets as *homogeneous* objects, without distinctions based on dimension.

Cartesian products of the form $A \times \{x\}$ can be taken as disjoint "equinumerous" copies of A, in accordance with the idea that singletons $\{x\}$ have a unitary numerosity.

(Unit)

- $\{x\} \approx \{x'\}$
- $A \times \{x\} \approx A$

Warning! When working is the general context of set theory, technical restrictions are needed.
E.g. if $A = \{a, (a, a), (a, a, a), \ldots\}$ then $A \times \{a\} \subset A$, against the *Part-whole principle*.

By the above considerations, Cartesian product of sets:

$$A \times B = \bigcup_{b \in B} A \times \{b\}$$

formalize the idea of product of A and B as the "*disjoint union* of B-many copies of A."

(Product)

$$A \approx A' \;\&\; B \approx B' \implies A \times B \approx A' \times B'$$

Warning! Also in this case a (mild) technical restriction is necessary to avoid contradictions with the *Part-Whole principle*. Specifically, for the above we assume that $(A \times B) \cap (A \cup B) = \emptyset$ and $(A' \times B') \cap (A' \cup B') = \emptyset$.

Although strictly speaking the Cartesian product is neither *commutative* nor *associative*, the natural bijections between $A \times B$ and $B \times A$, and between $(A \times B) \times C$ and $A \times (B \times C)$, should be assumed as "congruences" that preserve numerosity, in accordance with the fourth Euclidean notion.

(Product Commutativity)

$$A \times B \approx B \times A$$

(Product Associativity)

$$(A \times B) \times C \approx A \times (B \times C)$$

Other "numerosity-preserving" transformations for Cartesian products can be considered naturally, *e.g.*, permutations of coordinates. (We omit this topic here.)

Equinumerosity

We are now ready to formalize a "Euclidean" notion of numerosity for *all* sets in a sufficiently large universe $\mathbb{U}$.

Universes that we shall consider:

- Finite sets.
- Sets $A \subseteq \mathbb{N}^d$ of tuples of natural numbers (point sets over $\mathbb{N}$).
- Sets $A \subseteq \mathbb{R}^d$ of tuples of real numbers (point sets over $\mathbb{R}$).
- Entire set-theoretic universes.

We now give a definition of equinumerosity that incorporates all the properties discussed so far.

Definition

Equinumerosity is an equivalence relation $\approx$ such that:

1. $A \approx A'$ & $B \approx B' \implies A \cup B \approx A' \cup B'$, whenever $A \cap B = A' \cap B' = \emptyset$
2. $A \approx A'$ & $B \approx B' \implies A \setminus B \approx A' \setminus B'$, whenever $A \supseteq B$ and $A' \supseteq B'$
3. Exactly one of the following three conditions holds:
   - $A \approx B$
   - $A' \approx B$ for some proper superset $A' \supsetneq A$
   - $A \approx B'$ for some proper superset $B' \supsetneq B$
4. $A \approx A'$ & $B \approx B' \implies A \times B \approx A' \times B'$
5. $A \times \{x\} \approx A$
6. $A \times B \approx B \times A$
7. $(A \times B) \times C \approx A \times (B \times C)$

As already mentioned, properties (4), (5), (6) and (7) are postulated under suitable set-theoretic restrictions to avoid contradictions with the *Part-Whole principle*.

- Let the set of equivalence classes $\mathcal{N} = \mathbb{U}/\approx$ be the set of numerosities.
- Let the canonical projection $\mathfrak{num} : \mathbb{U} \to \mathcal{N}$ be the numerosity function.

We assume that the "universe" $\mathbb{U}$ is large enough so that:

- $\mathbb{U}$ includes *finite sets* of every cardinality.
- For every $A, B \in \mathbb{U}$ there exist *disjoint* sets $A', B' \in \mathbb{U}$ such that $A' \approx A$ and $B' \approx B$.
- $\mathbb{U}$ is closed under finite subsets, unions, and Cartesian products.

Theorem

Let $\approx$ be an equinumerosity relation on a universe $\mathbb{U}$, and let $\mathfrak{num} : \mathbb{U} \to \mathcal{N}$ be the corresponding numerosity function. Then

- *there exists a unique sum operation $+$ on $\mathcal{N}$,*
- *there exists a unique product operation $\cdot$ on $\mathcal{N}$,*
- *there exists a unique linear order $<$ on $\mathcal{N}$,*

such that

1. $\mathfrak{num}(A) + \mathfrak{num}(B) = \mathfrak{num}(A \cup B)$ *whenever* $A \cap B = \emptyset$.
2. $\mathfrak{num}(A) \cdot \mathfrak{num}(B) = \mathfrak{num}(A \times B)$ *whenever* $(A \times B) \cap (A \cup B) = \emptyset$.
3. $\mathfrak{num}(A) < \mathfrak{num}(B) \iff A \prec B$.

The resulting structure on $\mathcal{N}$ is the non-negative part of a discretely ordered ring $(\mathcal{Z}, 0, 1, +, \cdot, <)$ where $0 = \mathfrak{num}(\emptyset)$ and $1 = \mathfrak{num}(\{x\})$ for every $x \in \mathbb{U}$.

- **Numerosities as determined by equinumerosity relations satisfy the five Euclid's Common Notions** (of course, if interpreted appropriately).

Moreover:

- If we identify every $n \in \mathbb{N}$ with $\underbrace{1 + \ldots + 1}_{n \text{ times}} \in \mathcal{N}$, then $\mathfrak{num}(A) = |A|$ for every finite set $A \neq \emptyset$.

- The set of numerosities $\mathcal{N}$ includes (an isomorphic copy of) the natural numbers $\mathbb{N}$ as an initial segment.

Numerosity functions

An alternative presentation is as follows:

Definition

A numerosity function is a function $\mathfrak{num} : \mathbb{U} \to \mathcal{Z}_{\geq 0}$ taking values into the non-negative part of a *discretely ordered ring* $(\mathcal{Z}, 0, 1, +, \cdot, <)$, and such that:

1. $\mathfrak{num}(A \cup B) = \mathfrak{num}(A) + \mathfrak{num}(B)$ whenever $A \cap B = \emptyset$
2. $\mathfrak{num}(A \times B) = \mathfrak{num}(A) \cdot \mathfrak{num}(B)$ whenever $(A \times B) \cap (A \cup B) = \emptyset$.
3. $\mathfrak{num}(\{x\}) = 1$ for all singletons $\{x\}$.

Clearly, if $\approx$ is an equinumerosity relation, then the corresponding numerosity function $\mathfrak{num}$ satisfies the properties of the previous definition.

However, we stress that the three properties of a numerosity function **do not** guarantee that the corresponding equivalence relation defined by setting

$$A \approx B \iff \mathfrak{num}(A) = \mathfrak{num}(B)$$

satisfies all the properties of an equinumerosity relation.

In fact, a relevant property is missing.

(Euclidean property)

$\mathfrak{num}(A) < \mathfrak{num}(B) \iff \mathfrak{num}(A') = \mathfrak{num}(B)$ for a proper superset $A' \supsetneq A$.

- The equivalence relation $\approx$ determined by a numerosity function $\mathfrak{num}$ is an equinumerosity relation if and only if it satisfies the **(Euclidean property)**.

Definition

Let us call Euclidean a numerosity function that satisfies the **(Euclidean property)**.

Finite cardinalities

The basic example of a numerosity function is given by finite cardinalities:

$$|\cdot| : \{\text{Finite Sets}\} \to \mathbb{N} \cup \{0\} = \mathbb{Z}_{\geq 0}$$

All properties of an *Euclidean numerosity* are satisfied.

Non-example: Cantorian cardinality

Note that *Cantorian cardinality* satisfies the three properties of a numerosity function:

1. $\mathfrak{num}(A \cup B) = \mathfrak{num}(A) + \mathfrak{num}(B)$ whenever $A \cap B = \emptyset$
2. $\mathfrak{num}(A \times B) = \mathfrak{num}(A) \cdot \mathfrak{num}(B)$
3. $\mathfrak{num}(\{x\}) = 1$ for all singletons $\{x\}$.

It also satisfies the *Euclidean property*. However, *Cantorian cardinality* is <u>not</u> a numerosity function, because cardinal numbers <u>do not</u> have the structure of an ordered ring.

- For all infinite cardinals, $\kappa + \nu = \kappa \cdot \nu = \max\{\kappa, \nu\}$.

As a consequence, **(Subtraction)** and **(Part-Whole)** fail badly.

Numerosities on universes

- Are there numerosity functions that are also defined on infinite sets?

Numerosities on universes

- Are there numerosity functions that are also defined on infinite sets?

- **Yes!**
  In fact, there are numerosities on every mathematical universe.

By a *mathematical universe* we mean a suitable large collection of sets that includes the natural numbers and it is closed under all fundamental set-theoretic constructions.

Numerosity functions on entire set-theoretic universes do exist.

Theorem (ZFC)

For every mathematical universe $\mathbb{U}$ there exist numerosity functions $\mathfrak{num} : \mathbb{U} \to {}^*\mathbb{N}$ taking values into a set ${}^*\mathbb{N}$ of nonstandard natural numbers and are consistent with Cantorian cardinality:

$$\mathfrak{num}(A) = \mathfrak{num}(B) \implies |A| = |B|$$

- Does this result fully solve the problem we posed at the beginning of the talk?

QUESTION:

- Is the Cantorian notion of *cardinality* the only way to generalize (in a coherent manner) the notion of size of finite sets to infinite sets?
- In particular, can Cantorian cardinality be refined in order to save the Part-Whole principle?
- In particular, can Cantorian cardinality can be refined so that the resulting "numerosities" generalize the natural numbers while maintaining their nice algebraic properties?

Unfortunately not quite, because the *Euclidean Property* is missing.

(Euclidean property)

$\mathfrak{num}(A) < \mathfrak{num}(B) \iff \mathfrak{num}(A') = \mathfrak{num}(B)$ for a proper superset $A' \supsetneq A$.

Zermelian property

An equivalent reformulation of the Euclidean property of a numerosity function is the following:

(Zermelian property)

For all A, B, exactly one of the following holds:

- $\mathfrak{num}(A) = \mathfrak{num}(B)$.
- $\mathfrak{num}(A') = \mathfrak{num}(B)$ for some proper superset $A' \supsetneq A$.
- $\mathfrak{num}(A) = \mathfrak{num}(B')$ for some proper superset $B' \supsetneq B$.

Remark: The above *trichotomic property* follows directly from the axioms of equinumerosity, but it does not follow from the axioms of a *numerosity function*.

- In Zermelo-Fraenkel set theory ZF, the Zermelian property for cardinalities is equivalent to the axiom of choice.

It was Zermelo (1894) who introduced the *axiom of choice* to show that all sets are *well-orderable*. Consequently, given any pair of sets, one is equipotent to a subset of the other.

Before discussing the problem of their existence, let us observe that numbers of Euclidean numerosity functions satisfy really nice algebraic properties.

Field of fractions of numerosities

Proposition

Let $\mathfrak{num}$ *be a Euclidean numerosity. Then the fractions*

$$\mathfrak{F} = \left\{ \pm \frac{\mathfrak{num}(A)}{\mathfrak{num}(B)} \;\middle|\; B \neq \emptyset \right\}$$

form a non-Archimedean field.

PRLEM: The *Euclidean property* turns out really difficult to be realized. And even more problematic is its “subset” formulation.

(Euclidean property for subsets)

$\mathfrak{num}(A) < \mathfrak{num}(B) \iff \mathfrak{num}(A) = \mathfrak{num}(B')$ for a proper subset $B' \subsetneq B$.

We now focus on a special, but natural, class of universes.

Definition

By a space of point sets we mean the family of all *finite dimensional point sets* over a given line $\mathcal{L}$:

$$\mathbb{P}(\mathcal{L}) = \bigcup_{d=1}^{\infty} \{A \mid A \subseteq \mathcal{L}^d\} = \bigcup_{d=1}^{\infty} \mathcal{P}(\mathcal{L}^d).$$

- QUESTION: Are there *Euclidean* numerosities on a space of point sets?

Point sets over the natural numbers $\mathbb{N}$

Let us start with the natural numbers $\mathcal{L} = \mathbb{N}$.

A natural way of approximating a subset $A \subseteq \mathbb{N}$ by finite sets is considering its intersections with initial segments $\{1, \ldots, n\}$

Definition

A numerosity $\mathfrak{num}$ on the space $\mathbb{P}(\mathbb{N}) = \bigcup_{d=1}^{\infty} \mathcal{P}(\mathbb{N}^d)$ of point sets over $\mathbb{N}$ is asymptotic if for all $A, B \subseteq \mathbb{N}^d$:

$$|A \cap \{1, \ldots, n\}^d| \leq |B \cap \{1, \ldots, n\}^d| \text{ for all } n \implies \mathfrak{num}(A) \leq \mathfrak{num}(B)$$

Theorem (ZFC)

There exist asymptotic numerosities on the space of point sets $\mathbb{P}(\mathbb{N}) = \bigcup_{d=1}^{\infty} \mathcal{P}(\mathbb{N}^d)$ taking values into a set of hyper-natural numbers $^*\mathbb{N}$ of nonstandard analysis.

The possibility of obtaining the Euclidean property raises foundational issues.

Theorem (Blass - DN - Forti)

The existence of asymptotic numerosities on $\mathbb{P}(\mathbb{N})$ that satisfy the Euclidean property is independent of ZFC. Indeed, it is equivalent to the existence of quasi-selective ultrafilters on $\mathbb{N}$.

So, similarly to the continuum hypothesis, the existence of such *Euclidean asymptotic numerosities* on $\mathbb{P}(\mathbb{N})$ cannot be proved nor disproved by the usual principles of mathematics.

Numerosities and nonstandard analysis

As mentioned earlier, one can construct numerosity functions on the space of point sets $\mathbb{P}(\mathbb{N}) = \bigcup_{d=1}^{\infty} \mathcal{P}(\mathbb{N}^d)$ that take values into a set of hyper-natural numbers ${}^*\mathbb{N}$ of nonstandard analysis.

Conversely, models of Euclidean asymptotic numerosities are necessarily obtained by considering models of nonstandard analysis.

Theorem (Blass - DN -Forti)

Let $\mathfrak{num} : \mathbb{P}(\mathbb{N}) \twoheadrightarrow \mathcal{N}$ be an asymptotic numerosity that satisfies the "proper subset" Euclidean property. Then $\mathcal{N} \cong {}^*\mathbb{N} \cup \{0\}$ is a set of hyper-natural numbers of nonstandard analysis (in fact, isomorphic to an initial segment of an ultrapower of $\mathbb{N} \cup \{0\}$).

Hyper-natural and hyper-rational numbers

- What are the hyper-natural numbers?

The hyper-natural numbers ${}^*\mathbb{N}$ of nonstandard analysis are the positive part of a "special" *discretely ordered ring*, namely the ring of hyper-integers ${}^*\mathbb{Z}$:

$$ {}^*\mathbb{N} \;=\; \{\underbrace{1, 2, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}}\} $$

The hyper-rational numbers ${}^*\mathbb{Q}$ are the corresponding *ordered field* of fractions:

$$ {}^*\mathbb{Q} \;=\; \left\{ \frac{\nu}{N} \,\middle|\, \nu \in {}^*\mathbb{Z}, N \in {}^*\mathbb{N} \right\} $$

The hyper-rational field ${}^*\mathbb{Q}$ is *non*-Archimedean, in that it contains:

- Infinitesimal numbers $\varepsilon \neq 0$ such that

$$-\frac{1}{n} < \varepsilon < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

- Infinite numbers Ω such that:

$$|\Omega| > n \quad \text{for all } n \in \mathbb{N}.$$

One has a consistent notion of "small" (infinitesimal) and of "large" (infinite) number.

Definition

Two hyperrational numbers $\xi \approx \zeta$ are infinitely close when their distance $|\xi - \zeta|$ is infinitesimal.

Remark. The two structures $\mathbb{N}$ and ${}^*\mathbb{N}$ (and similarly $\mathbb{Q}$ and ${}^*\mathbb{Q}$) cannot be distinguished by any "elementary property" (first-order property).

Calculus by nonstandard analysis is grounded on the hyper-real field ${}^*\mathbb{R}$.

Some examples

Let us now see some simple examples of asymptotic numerosities of sets of natural numbers. Consider the following sets:

$$\mathbb{E} = \{n \in \mathbb{N} \mid n \text{ is even}\}; \quad \mathbb{O} = \{n \in \mathbb{N} \mid n \text{ is odd}\}.$$

- $\mathfrak{num}(\mathbb{E}) + \mathfrak{num}(\mathbb{O}) = \mathfrak{num}(\mathbb{N})$.

Moreover, by the asymptotic property, we have

- Either $\mathfrak{num}(\mathbb{E}) = \mathfrak{num}(\mathbb{O}) = \frac{\mathfrak{num}(\mathbb{N})}{2}$ if $\mathfrak{num}(\mathbb{N})$ is even; or
- $\mathfrak{num}(\mathbb{E}) = \frac{\mathfrak{num}(\mathbb{N})-1}{2}$ and $\mathfrak{num}(\mathbb{O}) = \frac{\mathfrak{num}(\mathbb{N})+1}{2}$ is α is odd.

For simplicity one can consistently postulates the following:

- For every k, all remainder classes mod k have equal numerosity, *i.e.* $\alpha = \mathfrak{num}(\mathbb{N})$ is a multiple of k and $\mathfrak{num}(\{n \in \mathbb{N} \mid n \equiv i \mod k\}) = \frac{\alpha}{k}$ for every $i = 0, \ldots, k-1$.

Besides, by using the *Euclidean property* that guarantees that the set of numerosities $\mathcal{N} = {}^*\mathbb{N} \cup \{0\}$ is a set of nonstandard natural numbers, one also obtains the following:

- $\mathfrak{num}(\{n^2 \mid n \in \mathbb{N}\}) = \lfloor \sqrt{\mathfrak{num}(\mathbb{N})} \rfloor$.
- $\mathfrak{num}(\{n^3 \mid n \in \mathbb{N}\}) = \lfloor \sqrt[3]{\mathfrak{num}(\mathbb{N})} \rfloor$.
- *and so forth.*

Again, one can consistently postulates the following:

- For every k, the numerosity $\alpha = \mathfrak{num}(\mathbb{N})$ is a power of k for every $k \in \mathbb{N}$, and $\mathfrak{num}(\{n^k \mid n \in \mathbb{N}\}) = \sqrt[k]{\alpha}$.

From discrete to continuum

With asymptotic Euclidean numerosities, one can also obtain the following:

Theorem

For every real number r there exist sets $A, B \subseteq \mathbb{N}$ such that

$$\frac{\mathfrak{num}(A)}{\mathfrak{num}(B)} \approx r.$$

For example:

$$\frac{\mathfrak{num}(\{1^2, 2^2, 3^2, \ldots, n^2, \ldots\})}{\mathfrak{num}(\{2 \cdot 1^2, 2 \cdot 2^2, 2 \cdot 3^2, \ldots, 2 \cdot n^2, \ldots\})} \approx \sqrt{2}.$$

In retrospect, the fact that the continuum of real numbers can be recovered from the discrete setting of *Euclidean asymptotic numerosities* is not surprising due to the following known property:

- If ${}^*\mathbb{N}$ is a set of hyper-natural numbers, then the field of fractions generated by ${}^*\mathbb{N}$ is a set ${}^*\mathbb{Q}$ of hyper-rational numbers of nonstandard analysis.

$${}^*\mathbb{Q} = \left\{ \pm\frac{\nu}{N} \;\middle|\; \nu, N \in {}^*\mathbb{N} \right\}.$$

- It is a known fact that the quotient of the *finite* hyper-rational numbers ${}^*\mathbb{Q}_{fin}$ modulo infinitesimal distance is isomorphic to the real line

$${}^*\mathbb{Q}_{fin}/\!\approx\; \cong\; \mathbb{R}$$

(In fact, here we need the fact that ${}^*\mathbb{N}$ is $\aleph_1$-saturated; for example, this is the case when ${}^*\mathbb{N}$ is an ultrapower of $\mathbb{N}$ over an incomplete countable ultrafilter.)

Point sets over the real line $\mathbb{R}$

Theorem (DN - Forti)

Assume that:

- *The size of the continuum* $\mathfrak{c} < \aleph_\omega$.
- *There exists a selective ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$ *(This is independent of ZFC).*

Then there exists a numerosity function $\mathfrak{num}$ *on the family* $\mathbb{P}(\mathbb{R}) = \bigcup_{d=1}^{\infty} \mathcal{P}(\mathbb{R}^d)$ *of point sets over the real line such that:*

- $\mathfrak{num}$ *takes values in a set* ${}^*\mathbb{N} \cup \{0\}$ *of hyper-natural numbers of nonstandard analysis.*
- $\mathfrak{num}$ *satisfies the Euclidean property when restricted to countable subsets.*
- $\mathfrak{num}$ *is coherent with Cantorian cardinality:* $\mathfrak{num}(A) = \mathfrak{num}(B) \Longrightarrow |A| = |B|$.

Universal numerosities

Definition

By a universal numerosity we mean a numerosity $\mathfrak{num}$ that is defined on a *mathematical universe* $\mathbb{U}$.

We already saw that *universal numerosity functions* exist on every mathematical universe $\mathbb{U}$. But what about the *Euclidean property*?

Theorem (DN - Forti)

Assume the axiom of constructibility $\mathbb{V} = L$. Then there exists a numerosity function $\mathfrak{num}$ on the universal class $\mathbb{V}$ of all sets such that:

- $\mathfrak{num}$ *satisfies the Euclidean property when restricted to countable subsets.*
- $\mathfrak{num}$ *is coherent with Cantorian cardinality:* $\mathfrak{num}(A) = \mathfrak{num}(B) \Rightarrow |A| = |B|$.

Numerosities and measures

The most basic way in mathematics to estimate the size of a set is that of *cardinality*. In a continuous setting, instead, one uses the notion of *measure*, which is a fundamental tool of analysis.

Definition

A finitely additive measure is a function μ defined on an algebra of sets, which takes non-negative real numbers as values (possibly taking also the value $+\infty$), and which satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$.

An *algebra of sets* $\mathfrak{A}$ over a set Ω a nonempty family of subsets of Ω which is closed under finite unions and intersections, and under relative complements, i.e. $A, B \in \mathfrak{A} \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathfrak{A}$.

Usually, in analysis one considers the stronger property of σ-additivity, which extends *finite additivity* to countable families:

- If $\{A_n\}$ is a countable family of pairwise disjoint sets then $\mu(\bigcup_n A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

The measure μ is called non-atomic when all finite sets in $\mathfrak{A}$ have measure zero, and this is the class of measures that are typically considered.

In a discrete setting, the properties of f.a.m. are satisfied by *finite cardinalities*, with the additional feature that singletons have size 1.

More generally, we have seen that such properties can be extended to infinite sets by using the notion of numerosity.

Having to do with measures, here we are not interested in the products, and so we arrive at the following "simplified" notion:

Definition

An elementary numerosity on a set Ω is a function

$$\mathfrak{num} : \mathcal{P}(\Omega) \to [0, \infty)_{\mathbb{F}}$$

defined on *all* subsets of Ω, taking values in the nonnegative part of an ordered field $\mathbb{F} \supseteq \mathbb{R}$, and such that the following two conditions are satisfied:

1. $\mathfrak{num}(\{x\}) = 1$ for every point $x \in \Omega$;
2. $\mathfrak{num}(A \cup B) = \mathfrak{num}(A) + \mathfrak{num}(B)$ whenever $A \cap B = \emptyset$.

Remark

If one takes $\mathbb{F} = \mathbb{R}$ then an elementary numerosity *num* exist on a set Ω if only if Ω is finite and $\mathfrak{num}(A) = |A|$ is the finite cardinality.

The two worlds of numerosities and measures are apparently very different. In fact, numerosities live in a discrete context that generalizes that of natural numbers; while measures are a typical tool of the continuum, and their use requires the *completeness property* of the real line in an essential way.

However, the two worlds can be connected!

Theorem

Let $(\Omega, \mathfrak{A}, \mu)$ be a non-atomic finitely additive measure on the infinite set Ω, and let $\mathfrak{B} \subseteq \mathfrak{A}$ be a subalgebra that does not contain nonempty sets of measure zero.
Then there exists an elementary numerosity $\mathfrak{num}$ on Ω such that:

1. *For all $B, B' \in \mathfrak{B}$ of finite measure:*

$$\mu(B) = \mu(B') \iff \mathfrak{num}(B) = \mathfrak{num}(B')$$

2. *If Z is any set of measure 1 then for every $A \in \mathfrak{A}$:*

$$\mu(A) \approx \frac{\mathfrak{num}(A)}{\mathfrak{num}(Z)}$$

The Lebesgue measure

Let λ be the Lebesgue measure on $\mathbb{R}$, and consider the algebra $\mathfrak{B}$ given by the finite unions of intervals of the form

$$[a, b), \quad [a, +\infty), \quad (-\infty, b).$$

Notice that $\mathfrak{B}$ does not contain nonempty sets of measure 0.
The previous theorem guarantees the existence of an elementary numerosity $\mathfrak{num}$ defined on *all* subsets of $\mathbb{R}$ such that:

(1) $\lambda(A) \approx \frac{\mathfrak{num}(A)}{\mathfrak{num}([0,1))}$ for every Lebesgue-measurable set A.

(2) $\mathfrak{num}([a, a + \ell)) = \mathfrak{num}([b, b + \ell))$ for every a, b and for every length ℓ.

Moreover, by finite additivity of $\mathfrak{num}$, it is easily shown that for every $a \in \mathbb{R}$ and for all $n, m \in \mathbb{N}$:

(3) $\mathfrak{num}([a, a + \frac{n}{m})) = \frac{n}{m} \cdot \mathfrak{num}([a, a + 1)) = \frac{n}{m} \cdot \mathfrak{num}([0, 1))$.

Non-Archimedean Probability

The notion of elementary numerosity has interesting applications in probability, also from a foundational point of view.
(Work by V. Benci, L. Horsten and S. Wenmackers.)

The starting idea is to extend the basic probability in a finite sample space to the infinite case, using numerosities instead of finite cardinalities.

- Let Ω be any infinite sample space.
- Pick a suitable elementary numerosity $num : \mathcal{P}(\Omega) \to [0,1]_{\mathbb{F}}$ taking values into a *non-Archimedean field* $\mathbb{F} \supset \mathbb{R}$.
- Define the probability $p(E)$ of an event $E \subseteq \Omega$ as the ratio $p(E) = \frac{\text{num}(E)}{\text{num}(\Omega)}$.

Fair lottery on $\mathbb{N}$

Define the non-Archimedean probability $p(E)$ of an event $E \subseteq \mathbb{N}$ by letting

$$p(E) \;=\; \frac{\mathfrak{num}(E)}{\mathfrak{num}(\mathbb{N})}$$

where $\mathfrak{num}$ is an elementary numerosity on $\mathbb{N}$. Then:

1. $p(\emptyset) = 0$, $p(\mathbb{N}) = 1$;
2. $p(A \cup B) = p(A) + p(B)$ whenever $A \cap B = \emptyset$;

The basic property here is that

- $p(E) = 0$ if and only if $E = \emptyset$.

Only the impossible event has probability zero!

Numerosities and Ramsey properties

The construction of models for numerosity functions that satisfy the Euclidean property typically involves consideration of combinatorial properties related to Ramsey's Theorem.

Theorem (Ramsey 1928)

Let X be infinite. For every finite coloring of the (unordered) pairs $[X]^2 = C_1 \cup \ldots \cup C_r$, there exists an infinite homogeneous set H, i.e., all pairs $[H]^2 \subseteq C_i$ are monochromatic.

A strategy to construct models of numerosity functions on a universe $\mathbb{U}$ is to take as the set numerosities $\mathcal{N} = {}^*\mathbb{N} \cup \{0\}$ an ultrapower

$$ {}^*\mathbb{N} = \mathbb{N}^I/\mathcal{U} $$

where:

- $I = \mathrm{Fin}(\mathbb{U})$ is the family of all finite subsets of $\mathbb{U}$.
- $\mathcal{U}$ is a suitable ultrafilter on I.
- For every $A \in \mathbb{U}$, its counting function $\varphi_A : I \to \mathbb{N}$ is defined by letting $\varphi_A(F) := |A \cap F|$.
- The numerosity function $\mathfrak{num} : \mathbb{U} \to {}^*\mathbb{N}_0$ is defined by letting:

$$ \mathfrak{num}(A) := \langle \varphi_A(F) \mid F \in I \rangle_{\mathcal{U}} \in {}^*\mathbb{N} \cup \{0\}. $$

- How to obtain the Euclidean property?

Suppose that $\mathfrak{num}(A) < \mathfrak{num}(B)$, that is

$$\{F \in I \mid \varphi_A(F) < \varphi_B(F)\} \in \mathcal{U}$$

We look for a set C such that $num(A) + num(C) = num(B)$, *i.e.*

$$\{F \in I \mid \varphi_A(F) + \varphi_C(F) = \varphi_B(F)\} \in \mathcal{U}.$$

- For every set C, its counting function $\varphi_C : I \to \mathbb{N} \cup \{0\}$ is non-decreasing, that is

$$F \subseteq G \implies \varphi_C(F) = |A \cap F| \leq |A \cap G| = \varphi_C(G).$$

So, a necessary condition to find such a set C is the following:

($\star$) If $\{F \in I \mid \varphi_A(F) < \varphi_B(F)\} \in \mathcal{U}$, then there exists a non-decreasing $\psi : I \to \mathbb{N}$ such that

$$\{F \in I \mid \varphi_A(F) + \psi(F) = \varphi_B(F)\} \in \mathcal{U}$$

Then, one can (hope to) construct a set C such that its counting function is such that $\varphi_C \equiv_{\mathcal{U}} \psi$, *i.e.* $\{F \in I \mid \varphi_C(F) = \psi(F)\} \in \mathcal{U}$.

The above property ($\star$) is obtained by taking an ultrafilter $\mathcal{U}$ with the following strong Ramsey property:

- Every function $\varphi : I \to \mathbb{N}$ is $\mathcal{U}$-equivalent to a non-decreasing function $\psi : I \to \mathbb{N}$.

We were eventually led to a Ramsey property for ultrafilters $\mathcal{U}$ on the finite parts $\mathrm{Fin}(X)$ of an arbitrary infinite set X.

Let $[\mathrm{Fin}(X)]^2_{\subset} = \{\{a \subset b\} \mid a, b \in \mathrm{Fin}(X)\}$ be the family of all comparable pairs of finite parts of X.

"Unbalanced Ramsey" ultrafilter $\mathcal{U}$ on $\mathrm{Fin}(X)$

Let $[\mathrm{Fin}(X)]^2_{\subset} = C_1 \cup C_2$ and assume that there are are no infinite chains $\{a_1 \subset a_2 \subset \ldots \subset a_n \subset a_{n+1} \subset \ldots\}$ that are homogeneous for C_1, *i.e.*, $\{a_i, a_j\} \in C_1$ for all $i < j$.
Then there exists a cofinal $H \in \mathcal{U}$ which is homogeneous for C_2, *i.e.* $\{a \subset b\} \in C_2$ for all comparable pairs $a \subset b$ of elements of H.

A set $H \subseteq \mathrm{Fin}(X)$ is cofinal if for every $a \in \mathrm{Fin}(X)$ there exists $b \in H$ with $b \supseteq a$.

Unbalanced Ramsey ultrafilters seem to represent a fundamental step toward the construction of Euclidean numerosities.

Let $\mathfrak{num}(A) < \mathfrak{num}(B)$, *i.e.*, $\{a \in \mathrm{Fin}(X) \mid |A \cap a| < |B \cap a|\} \in \mathcal{U}$.

Pick a positive function $\psi : \mathrm{Fin}(X) \to \mathbb{N}$ which is $\mathcal{U}$-equivalent to the function $a \longmapsto |B \cap a| - |A \cap a|$, and consider the partition $[\mathrm{Fin}(X)]^2_{\subset} = C_1 \cup C_2$ where:

- $C_1 = \{\{a \subset b\} \mid \psi(a) > \psi(b)\}$,
- $C_2 = \{\{a \subset b\} \mid \psi(a) \leq \psi(b)\}$.

Since there are no infinite homogeneous chains for C_1, the property of the unbalanced Ramsey ultrafilter implies the existence of a cofinal $H \in \mathcal{U}$ such that the restriction $\psi|_H : H \to \mathbb{N}$ is non-decreasing.

Having H to be cofinal is a necessary condition for constructing a set $C \subseteq X$ such that its counting function $a \mapsto |C \cap a|$ is $\mathcal{U}$-equivalent to ψ.

- Do "*unbalanced Ramsey ultrafilters*" on $\mathrm{Fin}(X)$ exist?

Joint work with M. Forti – in progress

- Grounding on a combinatorial result by Jech and Shelah, one can construct in ZFC an unbalanced Ramsey ultrafilter on ω_1. Thus, assuming CH, we conjecture the existence Euclidean numerosity functions on the universe of point-sets $\mathbb{P}(\mathbb{R}) = \bigcup_{d=1}^{\infty} \mathcal{P}(\mathbb{R}^d)$ over the real numbers.
- **Set-theoretic conjecture:** *For every infinite cardinal κ there exist unbalanced Ramsey ultrafilters on Fin(κ).*

Consequently, one could have a tool for constructing numerosity functions on entire universes that satisfy the Euclidean property without restrictions.


The End